SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

November 2011

Assessment Task 1
Year 11

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question if full marks are to be awarded.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 100

- Attempt sections A – E.
- Start each **NEW** section in a separate answer booklet.
- Hand in your answers in 5 separate bundles:

Section A
Section B
Section C
Section D
Section E

Examiner: *R. Boros*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

START A NEW ANSWER BOOKLET

SECTION A [20 marks] **Marks**

For these 10 questions there is one correct answer per question. Write down in your answer booklet the question number and letter of your answer.

1. In approximately what time will a sum of money triple itself at 10% p.a. interest compounded yearly? **[1]**
   - A. 8 years
   - B. 11.5 years
   - C. 9 years
   - D. 10.5 years

2. $2ab - a^2 - b^2$ factorised is **[1]**
   - A. $(a-b)^2$
   - B. $(-a-b)^2$
   - C. $-(a-b)^2$
   - D. $-(a+b)^2$

3. What number must be added to the expression $a^2 - 12a$ to make it a perfect square? **[1]**
   - A. -36
   - B. 24
   - C. -24
   - D. 36

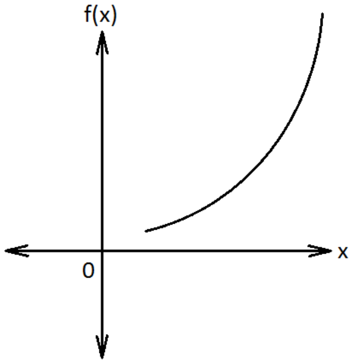
4. If $f(x) = 2x^2 + x$ and $g(x) = x^2 - 2x$, find the value of **[1]**
   $$2f(x) + g(x)$$
   - A. $5x^2 - x$
   - B. $5x^2$
   - C. $6x^2$
   - D. $3x^2 - x$

5. The discriminant of $x^2 - 2x + 3 = 0$ is **[1]**
   - A. -8
   - B. 8
   - C. -16
   - D. 16

6. Which of the following functions will give a straight line graph **[1]**
   - A. $y = \frac{x}{3}$
   - B. $y = \frac{1}{3x}$
   - C. $y = x(x-3)$
   - D. $y = \frac{x^3+2x+4}{3x}$

7. If $81^{2x+3} = 243^{5-x}$ then x equals **[1]**
   - A. -1
   - B. 0
   - C. 1
   - D. $\frac{13}{3}$

8. Given this part of the curve $y = f(x)$ **[1]**



A. $y' > 0\ y" < 0$
B. $y' > 0\ y" > 0$
C. $y' < 0\ y" > 0$
D. $y' < 0\ y" < 0$

9. Given $E = av^3t$ where a and v are positive constants, then $\frac{dE}{dt}$ is **[1]**

A. $3av^2t$
B. Cannot be differentiated
C. 1
D. av^3

10. Evaluate **[1]**

$$\sum_{i=7}^{10}(i-1)(i+1)$$

A. 290
B. 250
C. 190
D. 100

End of Multiple Choice Section

11. Sketch the following on separate diagrams showing all necessary features. **[4]**

i. $y = |x+3|$
ii. $y = \sqrt{16 - x^2}$

12. Insert two terms between 7 and $23\frac{5}{8}$ so that the 4 terms are terms in a geometric series. **[3]**

13. Given that $A = \left[\frac{9}{5}\right]^3, B = \left[\frac{1}{25}\right]$ and $C = 81$ find the value of x and y if $\frac{A^2}{B^5C^3} = 3^x \times 5^y$. **[3]**

End of Section A

START A NEW ANSWER BOOKLET

SECTION B [20 marks] **Marks**

1. Determine whether the function $f(x) = \frac{x^2}{x^2+4}$ is ODD or EVEN or neither. Justify your answer. **[2]**

2. For what values of k will $3x^2 - kx + 12$ be always positive? **[3]**

3. By using an appropriate substitution or otherwise, find all the solutions to $(x^2 - 2x)^2 - 7(x^2 - 2x) - 8 = 0$. **[3]**

4. State the domain and range of the following functions **[5]**
   i. $y = x + \frac{1}{x}$
   ii. $y = x^2\sqrt{1+x}$

5. Find the region in the number plane simultaneously determined in the first quadrant by: **[4]**

$$x \geq 0$$
$$y \geq 0$$
$$y > \frac{1}{x}$$
$$x + y < 4$$

6. Using the "completion of the square method" solve the following quadratic: $x(2x+1) = 5$, approximating your answer(s) to 3 significant figures. **[3]**

End of Section B

START A NEW ANSWER BOOKLET

SECTION C [20 marks] **Marks**

1. Write down the equation of the quadratic whose roots are $1\frac{1}{2}$ and 4. Fully expand and simplify your answer. **[2]**

2. Find the values of A, B and C such that **[3]**
$$x^2 + x + 1 \equiv A(x-2)^2 + B(x-2) + C.$$

3. If α and β are the roots of the equation $x^2 - 2x + 5 = 0$ find the values of **[5]**

    i. $\alpha + \beta$

    ii. $\alpha\beta$

    iii. $\alpha^2 + \beta^2$

    iv. $\frac{1}{\alpha} + \frac{1}{\beta}$

    v. $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

4. $P(12,18)$ is a point on the parabola $x^2 = 8y$. S is the focus of the parabola. The perpendicular through S to the line SP intersects the tangent line at P in M. Given that the tangent equation at P is $y = 3x - 18$, find **[5]**

    i. The value of the focal length

    ii. The coordinates of the focus S

    iii. The gradient of the line SP

    iv. The equation of the line SM written in gradient-intercept form

    v. The coordinates of the point M

5. Express the following in index form: $5\log x + \frac{1}{2}\log y = \log z$. **[2]**

6. Find from first principles, the gradient of the tangent to the curve $y = 3x^2 - 4x - 7$ at the point where $x = 1$. **[3]**

End of Section C

START A NEW ANSWER BOOKLET

SECTION D [20 marks] **Marks**

1. Differentiate with respect to x **[6]**

   i. $\frac{1}{3x^3}$

   ii. $(1-2x)^4$

   iii. $\frac{7}{(5-3x)^2}$

   iv. $(x^2+4)(x^4+3)$

2. Find the equation written in general form of the normal to the curve $y=\frac{2}{x+1}$ at the point (1,1). **[3]**

3. At what x values on the curve $y=x^3-2x^2+1$ is the tangent line parallel to the line $x+y=2$? **[3]**

4. For what values of x is $f(x)=\frac{1}{3}x^3-4x+7$ monotonic decreasing? **[2]**

5. For the curve $f(x)=\frac{1}{3}x^3-x^2-3x-6$, find the: **[6]**

   i. relative maximum and minimum stationary points.

   ii. absolute maximum and minimum points if $-2 \leq x \leq 6$.

   iii. the point of inflexion.

   iv. Hence or otherwise sketch the curve in the domain $-2 \leq x \leq 6$.

End of Section D

START A NEW ANSWER BOOKLET

SECTION E [20 marks] **Marks**

1. From 1 to 500 inclusive, find the sum of all the numbers which are divisible by 6 or 9. **[4]**

2. Yolanda decides to set up a trust fund for her grandson Benny. She invests $80 at the beginning of each month. The money is invested at 6% p.a. compounded monthly. The trust fund matures at the end of the month of her final investment, 25 years (300 investments later) after her first investment. **[5]**

   i. After 25 years, what will be the value of her first $80 invested? (correct to the nearest cent).

   ii. By writing a geometric series for the value of all Yolanda's investments, calculate the final value of Benny's trust fund (correct to the nearest cent).

3. A point $P(x, y)$ moves so that it is equidistant from the point $(0,1)$ and from the line $y = -1$. Derive the equation of the locus of P. **[2]**

4. Ryde Council borrowed $3 000 000 at the beginning of 2011 to fund a pilot stormwater catchment scheme. The annual interest rate is 12% p.a. Each year, interest is calculated on the balance at the beginning of the year and added to the balance owing. The debt is to be repaid by equal annual repayments of $480 000 with the first repayment being made at the end of 2011. **[5]**

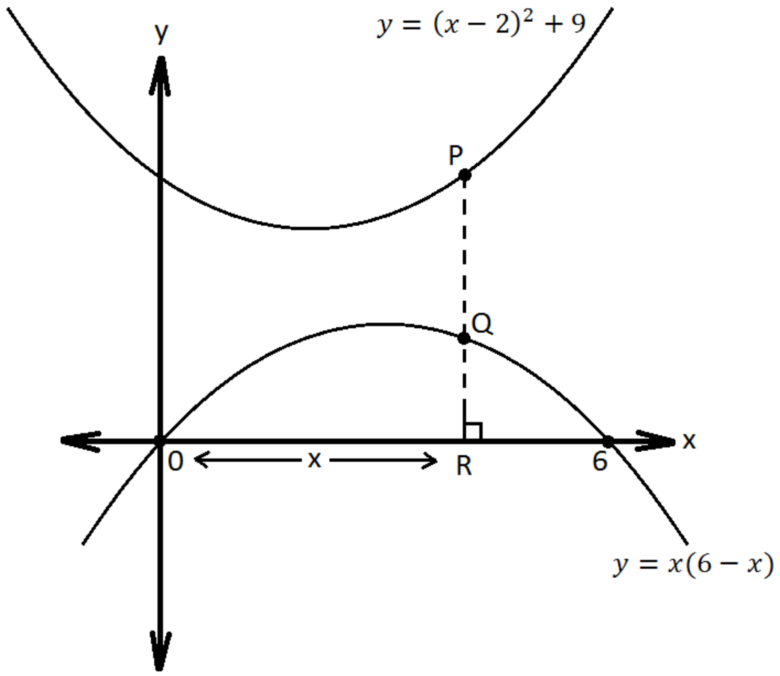
   If A_n is the balance owing after the n^{th} repayment, prove that:

   i. $A_2 = 3\,000\,000 \times 1.12^2 - 480\,000(1 + 1.12)$

   ii. $A_n = 1\,000\,000[4 - (1.12)^n]$

   iii. In which year (e.g. 9th, 10th, 11th etc) will Ryde Council make the final payment?

5. **[4]**



In the above figure, P is on the curve $y = (x-2)^2 + 9$ and Q is on the curve $y = x(6-x)$.

If the length of PQ is L units, find the minimum length of L and the value of x for which occurs.

End of Section E

End of Exam

2U TASK 1 - 2011

QUESTION ONE

(1) B

(2) C

(3) D

(4) B

(5) A

(6) A

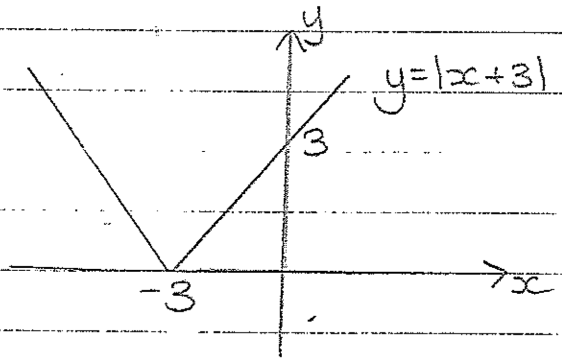
(7) C

(8) B

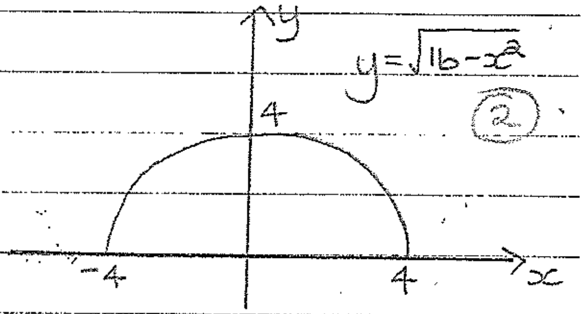
(9) D

(10) A

(11)



(2)



(2)

(12) $7, \ 7x, \ 7x^2, \ 23\frac{5}{8}$

$7x^3 = 23\frac{5}{8}$ (1)

$7x^3 = \frac{189}{8}$

$x^3 = \frac{27}{8}$

$x = \frac{3}{2}$ (1)

$7, \ \boxed{10\frac{1}{2}}, \ \boxed{15\frac{3}{4}}, \ 23\frac{5}{8}$ (1)

(13) $A = \left(\frac{9}{5}\right)^3 \quad B = \frac{1}{25} \quad C = 81$

$$\frac{A^2}{B^5 C^3} = \frac{\left(\frac{9}{5}\right)^6}{\left(\frac{1}{25}\right)^5 (81)^3} = \frac{3^{12}}{5^6} \div \left[\frac{1}{5^{10}} \times 3^{12}\right]$$ (3)

$$= \frac{3^{12}}{5^6} \times \frac{5^{10}}{3^{12}} = 5^4 \times 3^0$$

$x = 0 \qquad y = 4$

SECTION B:

1. $f(x) = \dfrac{x^2}{x^2+4}$

$f(-x) = \dfrac{(-x)^2}{(-x)^2+4}$

$= \dfrac{x^2}{x^2+4}$

$= f(x)$

$\therefore$ Even function

[2]

2. $3x^2 - kx + 12$ always positive

$\Rightarrow \quad \Delta < 0$

$\therefore (-k)^2 - 4 \times 3 \times 12 < 0$

$\therefore k^2 - 144 < 0$

$\therefore (k-12)(k+12) < 0$

$\therefore -12 < k < 12$

[3]

3. $(x^2-2x)^2 - 7(x^2-2x) - 8 = 0$

Let $u = x^2 - 2x$

$\therefore u^2 - 7u - 8 = 0$

$(u-8)(u+1) = 0$

$\therefore u = 8$ or $u = -1$

$\therefore x^2 - 2x = 8$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = 4$ or $x = -2$

$x^2 - 2x = -1$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$x = 1$

[3]

4(i) $y = x + \frac{1}{x}$.

Domain $x \neq 0$

For st pt $y' = 0$

$\therefore 1 - \frac{1}{x^2} = 0$

$\therefore x = \pm 1$

$\therefore y = \pm 2$

$\therefore$ Range $y \leq -2 \quad y \geq 2$

[3]

(ii) $y = x^2\sqrt{1+x}$

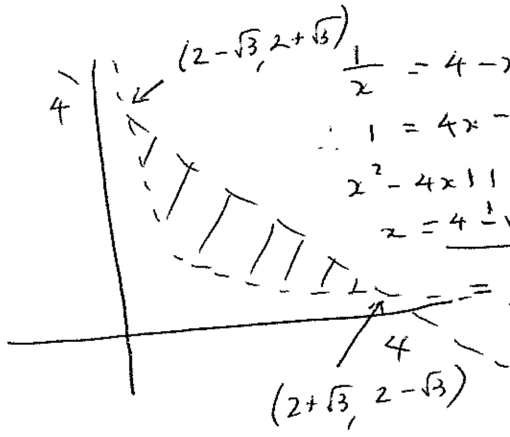
Domain $1+x \geq 0$

$\therefore x \geq -1$

Range $y \geq 0$

[2]

5.



$\frac{1}{x} = 4 - x$

$\therefore 1 = 4x - x^2$

$x^2 - 4x + 1 = 0$

$x = \dfrac{4 \pm \sqrt{16 - 4\times 1 \times 1}}{2}$

$= 2 \pm \sqrt{3}$.

[4]

6. $x(2x+1) = 5$

$2x^2 + x = 5$

$x^2 + \frac{1}{2}x = \frac{5}{2}$

$x^2 + \frac{1}{2}x + \frac{1}{16} = \frac{5}{2} + \frac{1}{16}$

$(x + \frac{1}{4})^2 = \frac{41}{16}$

$\therefore x + \frac{1}{4} = \pm\frac{\sqrt{41}}{4}$

$\therefore x = \dfrac{-1 \pm \sqrt{41}}{4}$

$\simeq 1.35, \quad -1.85$

[3]

Section C

1 $(x - 1\frac{1}{2})(x-4) = 0$

$(2x-3)(x-4) = 0$

$2x^2 - 11x + 12 = 0$

2 Given $x^2 + x + 1 \equiv A(x-2)^2 + B(x-2) + C$

$\therefore A = 1$, coefficient of x^2

Let $x=2$; $\therefore 2^2 + 2 + 1 \equiv 0 + 0 + C$

$C = 7$

Let $x=3$; $9 + 3 + 1 = 1(3-2)^2 + B(3-2) + 7$

$13 = 1 + B + 7$

$\therefore B = 5$

ie $A = 1$, $B = 5$ and $C = 7$

3 (i) $x^2 - 2x + 5 = 0$

$\alpha + \beta = \frac{-b}{a}$

$= 2$

(ii) $\alpha\beta = \frac{c}{a}$

$= 5$

(iii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= 2^2 - 2 \times 5$

$= -6$

(iv) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$

$= \frac{2}{5}$

(v) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$

$= \frac{-6}{5}$

4 $P(12, 18)$ on $x^2 = 8y$

$\therefore 4a = 8$

(i) $a = 2$

(ii) S, Focus is $(0, 2)$

(iii) Gradient of $PS = \frac{18-2}{12-0}$

$= \frac{4}{3}$

(iv) $\therefore$ Perpendicular to PS has gradient $-\frac{3}{4}$

$\therefore$ Eqn. of SM $y - 2 = \frac{-3}{4}(x-0)$

ie $3x + 4y - 8 = 0$

(v) Tangent at P is found to be $y = 3x - 18$ (may use formula)

Solve simultaneously $\begin{cases} y = 3x - 18 \\ 3x + 4y - 8 = 0 \end{cases}$

$3x + 4(3x - 18) - 8 = 0$

$15x - 80 = 0$

$x = \frac{16}{3}$

$y = -2$

M is $(\frac{16}{3}, -2)$

5 $5 \log x + \frac{1}{2} \log y = \log z$

$\log x^5 + \log y^{\frac{1}{2}} = \log z$

$\log x^5 y^{\frac{1}{2}} = \log z$

$x^5 y^{\frac{1}{2}} = z$

6 Let $f(x) = 3x^2 - 4x - 7$

now $f'(x) = \lim_{h\to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\}$

$= \lim_{h\to 0} \left\{ \frac{3(x+h)^2 - 4(x+h) - 7 - (3x^2 - 4x - 7)}{h} \right\}$

$= \lim_{h\to 0} \left\{ \frac{6xh + 3h^2 - 4h}{h} \right\}$

$= 6x - 4$

at $x = 1$ $f'(1) = 6 - 4 = \underline{2}$

Section D

1\. i. $y = \frac{1}{3}x^{-3}$

$y' = -x^{-4}$

$= -\frac{1}{x^4}$

ii. $y = (1-2x)^4$

$y' = 4(1-2x)^3 \cdot (-2)$

$= -8(1-2x)^3$

iii. $y = 7(5-3x)^{-2}$

$y' = -14(5-3x)^{-3} \cdot (-3)$

$= \frac{42}{(5-3x)^3}$

iv. $y = (x^2+4)(x^4+3)$

$= x^6 + 3x^2 + 4x^4 + 12$

$y' = 6x^5 + 16x^3 + 6x$

2\. $y = \frac{2}{x+1}$

$y = 2(x+1)^{-1}$

$y' = -2(x+1)^{-2} \cdot 1$

$= -\frac{2}{(x+1)^2}$

at $(1, 1)$

$m_T = -\frac{2}{(1+1)^2}$

$= -\frac{1}{2}$

$m_N = 2$

$y - y_1 = m(x - x_1)$

$y - 1 = 2(x - 1)$

$y - 1 = 2x - 2$

$2x - y - 1 = 0$

3. $y = x^3 - 2x^2 + 1$

$\underline{y' = 3x^2 - 4x}$

$x + y = 2$

$y = -x + 2$

$\underline{m = -1}$

$3x^2 - 4x = -1$

$3x^2 - 4x + 1 = 0$

$\frac{(3x-3)(3x-1)}{3} = 0$

$\times \mid 3$
$+ \mid -4$
$-3, -1$

$(x-1)(3x-1) = 0$

$x = 1, \frac{1}{3}$

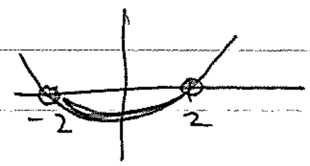
4. $f(x) = \frac{1}{3}x^3 - 4x + 7$

$f'(x) = x^2 - 4$

$f'(x) < 0$

$x^2 - 4 < 0$

$(x-2)(x+2) < 0$



$-2 < x < 2$

5. $f(x) = \frac{1}{3}x^3 - x^2 - 3x - 6$

$f'(x) = x^2 - 2x - 3$

$f''(x) = 2x - 2$

For stationary points $f'(x) = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

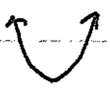
$x = 3, -1$

$f(3) = \frac{1}{3}(3)^3 - (3)^2 - 3(3) - 6$

$= -15$

$f''(3) = 2(3) - 2$

$= 4$



$f(-1) = \frac{1}{3}(-1)^3 - (-1)^2 - 3(-1) - 6$

$= -4\frac{1}{3}$

$f''(-1) = 2(-1) - 2$

$= -4$

$\therefore$ rel. min T.P @ $(3, -15)$

rel. max T.P @ $(-1, -4\frac{1}{3})$

endpoints of domain

$f(-2) = \frac{1}{3}(-2)^3 - (-2)^2 - 3(-2) - 6$

$= -6\frac{2}{3}$

$f(6) = \frac{1}{3}(6)^3 - (6)^2 - 3(6) - 6$

$= 12$

$\therefore$ absolute max. point is $(6, 12)$

absolute min. point is $(3, -15)$

For possible inflexion points $f''(x) = 0$

$2x - 2 = 0$

$2x = 2$

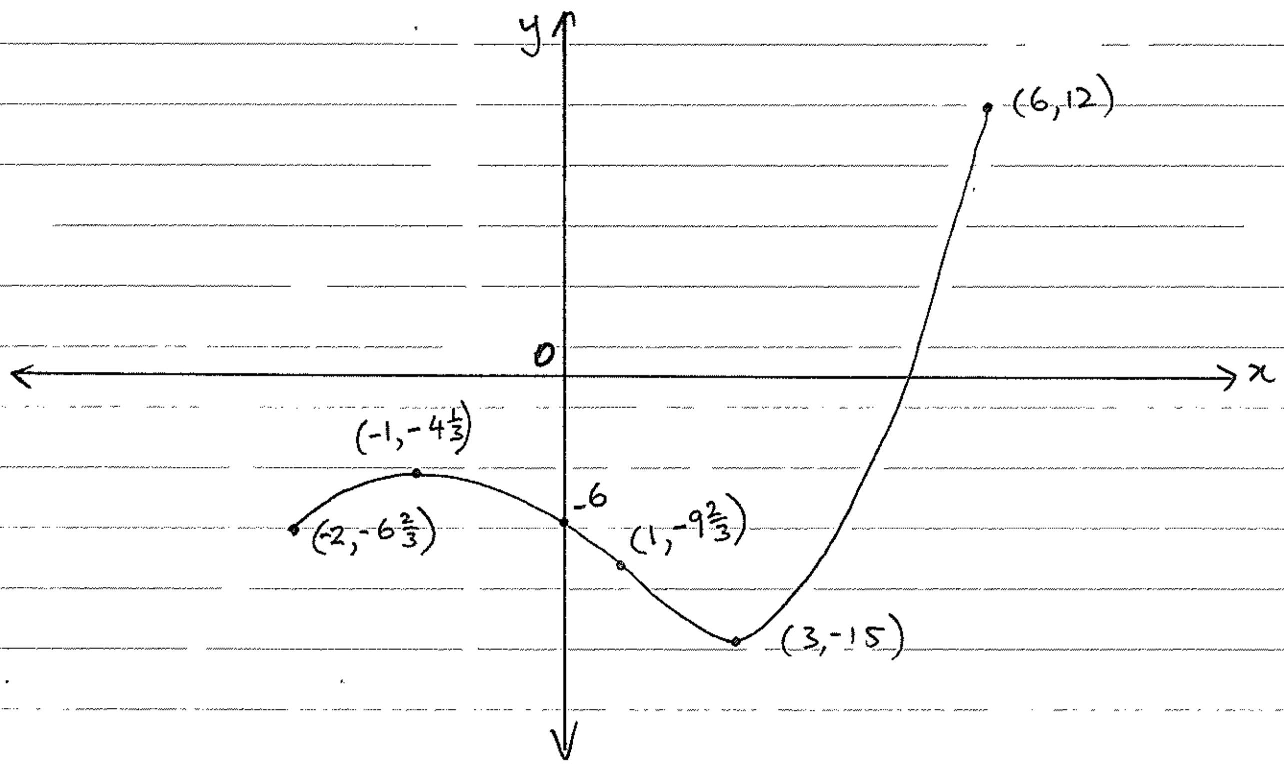
$x = 1$

$f(1) = \frac{1}{3}(1)^3 - (1)^2 - 3(1) - 6$

$= -9\frac{2}{3}$

Since $x = 1$ is between a rel. min & max T.P ($x = -1$, $x = 3$)

$(1, -9\frac{2}{3})$ is a point of inflexion



2011 Mathematics Assessment 1:
Solutions— Section E

1. From 1 to 500 inclusive, find the sum of all the numbers which are divisible by 6 or 9. [4]

Solution:

$$\begin{aligned} 500 \div 6 &= 83^1/_3, \quad \therefore 83 \text{ multiples of } 6 \\ 6 + 12 + 18 + \cdots + 6 \times 83 &= \frac{83}{2}(6 + 498) \\ &= 20\,916 \\ 500 \div 9 &= 55^5/_9, \quad \therefore 55 \text{ multiples of } 9 \\ 9 + 18 + 27 + \cdots + 9 \times 55 &= \frac{55}{2}(9 + 495) \\ &= 13\,860 \\ 500 \div 18 &= 27^7/_9, \quad \therefore 27 \text{ double-counted multiples of } 18 \\ 18 + 36 + 54 + \cdots + 18 \times 27 &= \frac{27}{2}(18 + 486) \\ &= 6804 \\ \therefore \text{ Sum is } 20\,916 + 13\,860 - 6804 &= 27\,972. \end{aligned}$$

2. Yolanda decides to set up a trust fund for her grandson, Benny. She invests \$80 at the beginning of each month. The money is invested at 6% p.a. compounded monthly. The trust fund matures at the end of the month of her final investment, 25 years (300 investments later) after her first investment. [5]

   (a) After 25 years, what will be the value of her first \$80 invested? (Correct to the nearest cent.)

   Solution: $\$80 \times \left(1 + \frac{6}{1200}\right)^{300} = \$357.20.$

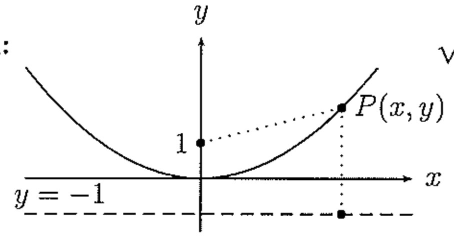
   (b) By writing a geometric series for the value of all Yolanda's investments, calculate the final value of Benny's trust fund (correct to the nearest cent).

   Solution:

$$\begin{aligned} \text{Total} &= \$80\left(1.005^{300} + 1.005^{299} + 1.005^{298} + \cdots + 1.005\right) \\ &= \frac{\$80 \times 1.005 \times \left(1.005^{300} - 1\right)}{1.005 - 1} \\ &= \$55\,716.71. \end{aligned}$$

3. A point $P(x, y)$ moves so that it is equidistant from the point $(0, 1)$ and from the line $y = -1$. Derive the equation of the locus of P. [2]

Solution:



$$\begin{aligned}\sqrt{(x-0)^2+(y-1)^2} &= y - {}^{-}1\\ x^2+y^2-2y+1 &= y^2+2y+1\\ x^2 &= 4y.\end{aligned}$$

4. Ryde Council borrowed \$3 000 000 at the beginning of 2011 to fund a pilot stormwater catchment scheme. The annual interest rate is 12% p.a. Each year, interest is calculated on the balance at the beginning of the year and added to the balance owing. The debt is to be repaid by equal annual repayments of \$480 000 with the first repayment being made at the end of 2011. [5]

If A_n is the balance owing after the n^{th} repayment, prove that

(a) $A_2 = 3\,000\,000 \times 1.12^2 - 480\,000(1 + 1.12)$

Solution:
$$\begin{aligned}A_0 &= 3\,000\,000\\ A_1 &= 3\,000\,000 \times 1.12 - 480\,000\\ A_2 &= (3\,000\,000 \times 1.12 - 480\,000) \times 1.12 - 480\,000\\ &= 3\,000\,000 \times 1.12^2 - 480\,000(1+1.12)\end{aligned}$$

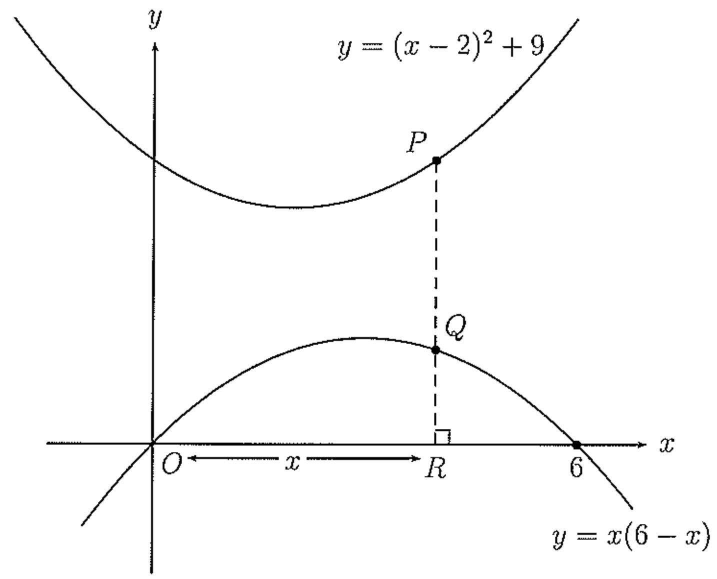
(b) $A_n = 1\,000\,000\big[4 - (1.12)^n\big]$

Solution:
$$\begin{aligned}A_3 &= \big(3\,000\,000 \times 1.12^2 - 480\,000(1+1.12)\big) \times 1.12 - 480\,000\\ &= 3\,000\,000 \times 1.12^3 - 480\,000(1+1.12+1.12^2)\end{aligned}$$
$$\begin{aligned}A_n &= 3\,000\,000 \times 1.12^n - 480\,000(1+1.12+1.12^2+\cdots+1.12^{n-1})\\ &= 3\,000\,000 \times 1.12^n - \frac{480\,000(1.12^n-1)}{1.12-1}\\ &= 3\,000\,000 \times 1.12^n - 4\,000\,000(1.12^n-1)\\ &= 4\,000\,000 - 1\,000\,000 \times 1.12^n\\ &= 1\,000\,000(4-1.12^n)\end{aligned}$$

(c) In which year (e.g. 9^{th}, 10^{th}, 11^{th} etc.) will Ryde Council make the final payment?

Solution:
$$\begin{aligned}1\,000\,000(4-1.12^n) &= 0\\ \therefore 1.12^n &= 4\\ n\ln 1.12 &= \ln 4\\ n &\approx 12.23\end{aligned}$$

So the last payment will be in the 13^{th} year.

5. 

4

In the above figure, P is on the curve $y = (x-2)^2 + 9$ and Q is on the curve $y = x(6-x)$.

If the length of PQ is ℓ units, find the minimum length of ℓ and the value of x for which it occurs.

Solution: Method 1—

$P\big(x,\ (x-2)^2+9\big),\ Q\big(x,\ x(6-x)\big)$

$$\begin{aligned} \ell &= x^2 - 4x + 13 - (6x - x^2) \\ &= 2x^2 - 10x + 13 \\ &= 2\left(x^2 - 5x + \tfrac{25}{4}\right) + 13 - \tfrac{25}{2} \\ &= 2\left(x - \tfrac{5}{2}\right)^2 + \tfrac{1}{2} \end{aligned}$$

$\therefore$ The minimum value of ℓ is $\frac{1}{2}$ when $x = \frac{5}{2}$.

Solution: Method 2—

$P\big(x,\ (x-2)^2+9\big),\ Q\big(x,\ x(6-x)\big)$

$$\begin{aligned} \ell &= x^2 - 4x + 13 - (6x - x^2) \\ &= 2x^2 - 10x + 13 \\ \frac{d\ell}{dx} &= 4x - 10 \\ &= 0 \text{ when } x = 2^1/_2 \\ \frac{d^2\ell}{dx^2} &= 4 \end{aligned}$$

$\therefore$ Minimum when $x = 2^1/_2$.

Minimum length is $2 \times 2^1/_2{}^2 - 10 \times 2^1/_2 + 13 = \dfrac{1}{2}$.